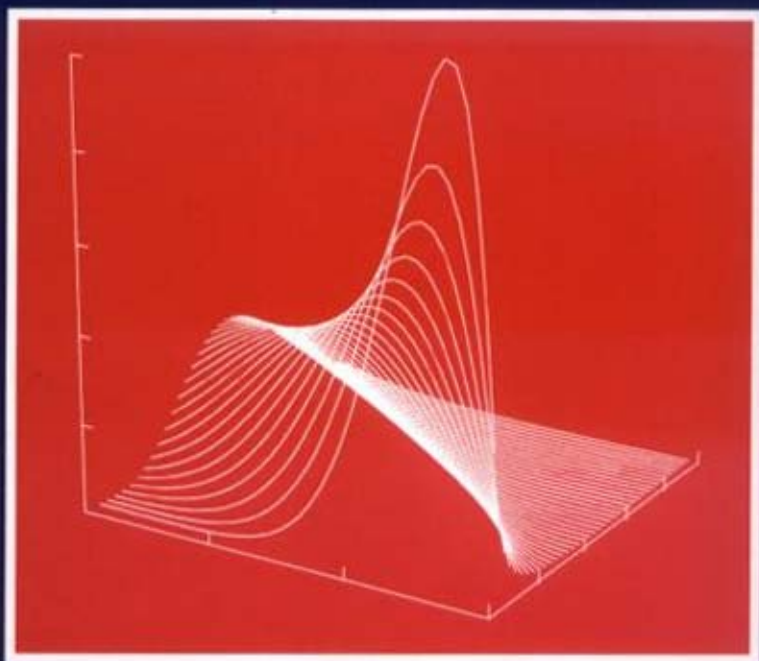


Advanced Calculus with Applications in Statistics

Second Edition



André I. Khuri

WILEY SERIES IN PROBABILITY AND STATISTICS

Advanced Calculus with
Applications in Statistics

Second Edition

Advanced Calculus with Applications in Statistics

Second Edition
Revised and Expanded

André I. Khuri

University of Florida
Gainesville, Florida



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To Ronnie, Marcus, and Roxanne
and
In memory of my sister Ninette

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Preface

This edition provides a rather substantial addition to the material covered in the first edition. The principal difference is the inclusion of three new chapters, Chapters 10, 11, and 12, in addition to an appendix of solutions to exercises.

Chapter 10 covers orthogonal polynomials, such as Legendre, Chebyshev, Jacobi, Laguerre, and Hermite polynomials, and discusses their applications in statistics. Chapter 11 provides a thorough coverage of Fourier series. The presentation is done in such a way that a reader with no prior knowledge of Fourier series can have a clear understanding of the theory underlying the subject. Several applications of Fourier series in statistics are presented. Chapter 12 deals with approximation of Riemann integrals. It gives an exposition of methods for approximating integrals, including those that are multidimensional. Applications of some of these methods in statistics are discussed. This subject area has recently gained prominence in several fields of science and engineering, and, in particular, Bayesian statistics. The material should be helpful to readers who may be interested in pursuing further studies in this area.

A significant addition is the inclusion of a major appendix that gives detailed solutions to the vast majority of the exercises in Chapters 1–12. This supplement was prepared in response to numerous suggestions by users of the first edition. The solutions should also be helpful in getting a better understanding of the various topics covered in the book.

In addition to the aforementioned material, several new exercises were added to some of the chapters in the first edition. Chapter 1 was expanded by the inclusion of some basic topological concepts. Chapter 9 was modified to accommodate Chapter 10. The changes in the remaining chapters, 2 through 8, are very minor. The general bibliography was updated.

The choice of the new chapters was motivated by the evolution of the field of statistics and the growing needs of statisticians for mathematical tools beyond the realm of advanced calculus. This is certainly true in topics concerning approximation of integrals and distribution functions, stochastic

processes, time series analysis, and the modeling of periodic response functions, to name just a few.

The book is self-contained. It can be used as a text for a two-semester course in advanced calculus and introductory mathematical analysis. Chapters 1–7 may be covered in one semester, and Chapters 8–12 in the other semester. With its coverage of a wide variety of topics, the book can also serve as a reference for statisticians, and others, who need an adequate knowledge of mathematics, but do not have the time to wade through the myriad mathematics books. It is hoped that the inclusion of a separate section on applications in statistics in every chapter will provide a good motivation for learning the material in the book. This represents a continuation of the practice followed in the first edition.

As with the first edition, the book is intended as much for mathematicians as for statisticians. It can easily be turned into a pure mathematics book by simply omitting the section on applications in statistics in a given chapter. Mathematicians, however, may find the sections on applications in statistics to be quite useful, particularly to mathematics students seeking an interdisciplinary major. Such a major is becoming increasingly popular in many circles. In addition, several topics are included here that are not usually found in a typical advanced calculus book, such as approximation of functions and integrals, Fourier series, and orthogonal polynomials. The fields of mathematics and statistics are becoming increasingly intertwined, making any separation of the two unpropitious. The book represents a manifestation of the interdependence of the two fields.

The mathematics background needed for this edition is the same as for the first edition. For readers interested in statistical applications, a background in introductory mathematical statistics will be helpful, but not absolutely essential. The annotated bibliography in each chapter can be consulted for additional readings.

I am grateful to all those who provided comments and helpful suggestions concerning the first edition, and to my wife Ronnie for her help and support.

ANDRÉ I. KHURI

Gainesville, Florida

Preface to the First Edition

The most remarkable mathematical achievement of the seventeenth century was the invention of calculus by Isaac Newton (1642–1727) and Gottfried Wilhelm Leibniz (1646–1716). It has since played a significant role in all fields of science, serving as its principal quantitative language. There is hardly any scientific discipline that does not require a good knowledge of calculus. The field of statistics is no exception.

Advanced calculus has had a fundamental and seminal role in the development of the basic theory underlying statistical methodology. With the rapid growth of statistics as a discipline, particularly in the last three decades, knowledge of advanced calculus has become imperative for understanding the recent advances in this field. Students as well as research workers in statistics are expected to have a certain level of mathematical sophistication in order to cope with the intricacies necessitated by the emerging of new statistical methodologies.

This book has two purposes. The first is to provide beginning graduate students in statistics with the basic concepts of advanced calculus. A high percentage of these students have undergraduate training in disciplines other than mathematics with only two or three introductory calculus courses. They are, in general, not adequately prepared to pursue an advanced graduate degree in statistics. This book is designed to fill the gaps in their mathematical training and equip them with the advanced calculus tools needed in their graduate work. It can also provide the basic prerequisites for more advanced courses in mathematics.

One salient feature of this book is the inclusion of a complete section in each chapter describing applications in statistics of the material given in the chapter. Furthermore, a large segment of Chapter 8 is devoted to the important problem of optimization in statistics. The purpose of these applications is to help motivate the learning of advanced calculus by showing its relevance in the field of statistics. There are many advanced calculus books designed for engineers or business majors, but there are none for statistics

majors. This is the first advanced calculus book to emphasize applications in statistics.

The scope of this book is not limited to serving the needs of statistics graduate students. Practicing statisticians can use it to sharpen their mathematical skills, or they may want to keep it as a handy reference for their research work. These individuals may be interested in the last three chapters, particularly Chapters 8 and 9, which include a large number of citations of statistical papers.

The second purpose of the book concerns mathematics majors. The book's thorough and rigorous coverage of advanced calculus makes it quite suitable as a text for juniors or seniors. Chapters 1 through 7 can be used for this purpose. The instructor may choose to omit the last section in each chapter, which pertains to statistical applications. Students may benefit, however, from the exposure to these additional applications. This is particularly true given that the trend today is to allow the undergraduate student to have a major in mathematics with a minor in some other discipline. In this respect, the book can be particularly useful to those mathematics students who may be interested in a minor in statistics.

Other features of this book include a detailed coverage of optimization techniques and their applications in statistics (Chapter 8), and an introduction to approximation theory (Chapter 9). In addition, an annotated bibliography is given at the end of each chapter. This bibliography can help direct the interested reader to other sources in mathematics and statistics that are relevant to the material in a given chapter. A general bibliography is provided at the end of the book. There are also many examples and exercises in mathematics and statistics in every chapter. The exercises are classified by discipline (mathematics and statistics) for the benefit of the student and the instructor.

The reader is assumed to have a mathematical background that is usually obtained in the freshman–sophomore calculus sequence. A prerequisite for understanding the statistical applications in the book is an introductory statistics course. Obviously, those not interested in such applications need not worry about this prerequisite. Readers who do not have any background in statistics, but are nevertheless interested in the application sections, can make use of the annotated bibliography in each chapter for additional reading.

The book contains nine chapters. Chapters 1–7 cover the main topics in advanced calculus, while chapters 8 and 9 include more specialized subject areas. More specifically, Chapter 1 introduces the basic elements of set theory. Chapter 2 presents some fundamental concepts concerning vector spaces and matrix algebra. The purpose of this chapter is to facilitate the understanding of the material in the remaining chapters, particularly, in Chapters 7 and 8. Chapter 3 discusses the concepts of limits and continuity of functions. The notion of differentiation is studied in Chapter 4. Chapter 5 covers the theory of infinite sequences and series. Integration of functions is

the theme of Chapter 6. Multidimensional calculus is introduced in Chapter 7. This chapter provides an extension of the concepts of limits, continuity, differentiation, and integration to functions of several variables (multivariable functions). Chapter 8 consists of two parts. The first part presents an overview of the various methods of optimization of multivariable functions whose optima cannot be obtained explicitly by standard advanced calculus techniques. The second part discusses a variety of topics of interest to statisticians. The common theme among these topics is optimization. Finally, Chapter 9 deals with the problem of approximation of continuous functions with polynomial and spline functions. This chapter is of interest to both mathematicians and statisticians and contains a wide variety of applications in statistics.

I am grateful to the University of Florida for granting me a sabbatical leave that made it possible for me to embark on the project of writing this book. I would also like to thank Professor Rocco Ballerini at the University of Florida for providing me with some of the exercises used in Chapters, 3, 4, 5, and 6.

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